

LINEAR DYNAMICS IN REPRODUCING KERNEL HILBERT SPACES

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ABSTRACT. Complementing the earlier results on dynamics of unilateral weighted shifts, we obtain a sufficient (but not necessary, with supporting examples) condition for hypercyclicity, mixing and chaos for M_z^* , the adjoint of M_z , on vector-valued analytic reproducing kernel Hilbert spaces \mathcal{H} in terms of the derivatives of kernel functions on the open unit disc \mathbb{D} in \mathbb{C} . Here M_z denotes the multiplication operator by the coordinate function z , that is

$$(M_z f)(w) = wf(w),$$

for all $f \in \mathcal{H}$ and $w \in \mathbb{D}$. We study the special case of quasi-scalar reproducing kernel Hilbert spaces and analyze hypercyclicity for sum of reproducing kernels in the sense of Aronsjajn. We also present a complete characterization of hypercyclicity of M_z^* on some special classes of vector-valued analytic reproducing kernel Hilbert spaces.

1. INTRODUCTION

Motivated by challenges in dynamics of bounded linear operators on Banach spaces, in this paper, we initiate the study of dynamics of adjoints of the multiplication operators on analytic reproducing kernel Hilbert spaces. Here we bring together two sets of ideas - the Hypercyclicity Criterion and analytic reproducing kernels on the open unit disc \mathbb{D} in \mathbb{C} .

More specifically, motivated by Salas' work [25] on characterization of hypercyclicity of backward weighted shift operators on l^p spaces, $1 \leq p < \infty$, we propose a general question on hypercyclicity of the adjoints of the multiplication operators on analytic reproducing kernel Hilbert spaces. In Theorems 4.1 and 4.2, we present sufficient conditions for testing hypercyclicity, topological mixing and chaoticity of adjoints of the multiplication operators on reproducing kernel Hilbert spaces in terms of derivatives of analytic kernel functions.

In the special case of scalar-valued analytic kernel functions, our main theorem on hypercyclicity states that (see Theorem 5.1): Let $k : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$ be an analytic kernel function and let $\mathcal{H}(k)$ denote the reproducing kernel Hilbert space corresponding to k (see Section 3 for definitions). Suppose that the multiplication operator M_z on $\mathcal{H}(k)$ is bounded. Then M_z^* on $\mathcal{H}(k)$ is hypercyclic if

$$\liminf_n \left(\frac{1}{(n!)^2} \frac{\partial^{2n} k}{\partial z^n \partial \bar{w}^n}(0, 0) \right) = 0.$$

This result unifies previous work on the sufficient condition for hypercyclicity of weighted shift operators (see Gethner and Shapiro [16], Kitai [20], Rolewicz [24], and Salas [25]).

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The sufficient condition for hypercyclicity in Theorem 4.1 is not necessary, in general (see the example in Subsection 5.3).

For the context of hypercyclicity of translation operators on reproducing kernel Hilbert spaces of entire functions, we refer the reader to Baranov [2], Chan and Shapiro [11], Garcia, Hernandez-Medina and Portal [15] and Godefroy and Shapiro [17]. Also see Bonet [8].

After a preliminary section devoted to fixing the notations and recalling the basic facts about dynamics of bounded linear operators, in Section 3, we describe the necessary background of reproducing kernel Hilbert spaces. Some of our results are less standard. Moreover, it is worth pointing out that the ‘‘backward shift’’ behavior of the adjoints of multiplication operators on analytic reproducing kernel Hilbert spaces in Theorem 3.3 is of independent interest.

Section 4 contains the main results on dynamics of M_z^* on analytic reproducing kernel Hilbert spaces. In Section 5, we provide an explicit example of an analytic reproducing kernel space on which M_z^* is hypercyclic but does not satisfy the sufficient condition for hypercyclicity in Theorem 4.1. We also relate the dynamics of M_z^* on sum of analytic reproducing kernel Hilbert spaces. The final section deals with the converse of Theorem 4.1 under different assumptions on analytic kernel functions.

2. PRELIMINARIES

In this section we fix some notation and recall definitions and facts on dynamics of bounded linear operators. For more details concerning linear dynamics, we refer the reader to the monographs by F. Bayart and E. Matheron [4] and K.G. Grosse-Erdmann and A. Peris [19].

Let X be a Banach space and let $\mathcal{B}(X)$ be the set of all bounded linear operators on X . Let T be a bounded linear operator on X . We say that T is:

(i) *hypercyclic* if there exists $x \in X$ such that

$$\{x, Tx, T^2x, \dots\}$$

is dense in X ,

(ii) *chaotic* if T is hypercyclic and it has dense set of periodic points (a vector y in X is called *periodic* for T if there exists $k \in \mathbb{N}$ such that $T^k y = y$), and

(iii) *topologically transitive*, if

$$\bigcup_{m=0}^{\infty} T^{-m}(U),$$

is dense in X for every non-empty open set U in X .

From the point of view of hypercyclic operators, in this paper, Banach spaces are always assumed to be separable.

A well known theorem of Birkhoff, known as Birkhoff’s transitivity theorem, states that (cf. [4]): T in $\mathcal{B}(X)$ is hypercyclic if and only if T is topologically transitive.

Another important notion in dynamical systems is the topological mixing: T in $\mathcal{B}(X)$ is said to be *topologically mixing* if, given any two non-empty open subsets U and V of X , there exists N in \mathbb{N} such that

$$T^{-j}(U) \cap V \neq \emptyset,$$

for all $j \geq N$.

The classical examples of hypercyclic operators are translation operators [7], differential operators [22] and commutators of translation operators [17]. The first example of a hypercyclic backward weighted shift was produced by Rolewicz [24]: for each $1 \leq p < \infty$ and $|\alpha| > 1$, the backward weighted shift αB is hypercyclic on l^p , where B on l^p is the unweighted backward shift:

$$B(\{c_0, c_1, c_2, \dots\}) = \{c_1, c_2, c_3, \dots\},$$

for all $\{c_0, c_1, c_2, \dots\} \in l^p$ (see also Kitai [20] and Gethner and Shapiro [16]). Salas [25] provides further improvement by characterizing hypercyclicity of backward weighted shifts: Let $\lambda = \{\lambda_n\}_{n \geq 1}$ be a bounded sequence of positive real numbers. Then the backward weighted shift B_λ on l^p , $1 \leq p < \infty$, defined by

$$B_\lambda(\{c_0, c_1, c_2, \dots\}) = \{\lambda_1 c_1, \lambda_2 c_2, \lambda_3 c_3, \dots\},$$

for all $\{c_0, c_1, c_2, \dots\} \in l^p$, is hypercyclic if and only if

$$\limsup_n \{\lambda_1 \lambda_2 \cdots \lambda_n\} = \infty.$$

For our purposes, we interpret the $p = 2$ case from function Hilbert space [26] point of view: Given a sequence of positive real numbers $\beta = \{\beta_n\}_{n \geq 0}$, we denote $H^2(\beta)$ for the Hilbert space of all formal power series

$$f(z) = \sum_{n \geq 0} a_n z^n,$$

such that

$$\|f\|_{H^2(\beta)}^2 := \sum_{n \geq 0} \frac{|a_n|^2}{\beta_n^2} < \infty.$$

Clearly, the set of functions $\{\beta_n z^n\}$ forms an orthonormal basis in $H^2(\beta)$. If, in addition, we assume that

$$\limsup_n \frac{\beta_{n+1}}{\beta_n} = 1,$$

then $H^2(\beta) \subseteq \mathcal{O}(\mathbb{D})$ (cf. Shields [26]), the space of all analytic functions on \mathbb{D} . It also follows that the multiplication operator M_z on $H^2(\beta)$ defined by

$$(M_z f)(w) = w f(w),$$

for all $f \in H^2(\beta)$ and $w \in \mathbb{D}$, is bounded if and only if

$$\sup_n \frac{\beta_n}{\beta_{n+1}} < \infty.$$

In this case, Salas' classification result can be interpreted as follows: M_z^* is hypercyclic on $H^2(\beta)$ if and only if

$$\liminf_n \beta_n = 0.$$

Moreover, the Costakis-Sambarino theorem [14] states that: M_z^* is mixing if and only if

$$\lim_n \beta_n = 0.$$

The well-known and useful sufficient conditions for hypercyclicity of operators on Banach spaces state that (see Kitai [20], Gethner and Shapiro [16], Godefroy and Shapiro [17], and Bes and Peris [6]): Let X be a Banach space, D be a dense set in X , and let

$T \in \mathcal{B}(X)$. If there exist a strictly increasing sequence $\{n_k\} \subseteq \mathbb{N}$ and a map $S : D \rightarrow D$ such that

- (i) $T^{n_k} S^{n_k} \rightarrow I_D$,
- (ii) $T^{n_k} \rightarrow 0$, and
- (iii) $S^{n_k} \rightarrow 0$,

pointwise on D , then T is hypercyclic. Moreover, if

$$n_k = k,$$

for all $k \in \mathbb{N}$, then T is topologically mixing.

The topological mixing part in the above result is due to Chen and Shaw (see (i) \Leftrightarrow (ii), Theorem 2.7, [12]).

For our purposes it is convenient to use the following version of the Hypercyclicity Criterion (cf. Definition 1.1 in [5] and also see [6]):

Theorem 2.1. (*The Hypercyclicity Criterion*) *Let X be a Banach space, D be a dense set in X , and let $T \in \mathcal{B}(X)$. Suppose that $\{n_k\} \subseteq \mathbb{N}$ is a strictly increasing sequence. If*

- (i) $T^{n_k} \rightarrow 0$ pointwise on D , and
- (ii) for each $f \in D$ there exists a sequence $\{f_k\}_{k \geq 1} \subseteq X$ such that

$$f_k \rightarrow 0 \quad \text{and} \quad T^{n_k} f_k \rightarrow f,$$

then T is hypercyclic. Moreover, if

$$n_k = k,$$

for all $k \in \mathbb{N}$, then T is topologically mixing.

We now proceed to a chaoticity criterion (see Bonilla and Grosse-Erdmann [10], page 386): Let T be an operator on a Banach space X , D be dense in X and $S : D \rightarrow D$ be map. If

- (i) $\sum_{n \geq 0} T^n x$ and $\sum_{n \geq 0} S^n x$ are unconditionally convergent, and
- (ii) $T S x = x$,

for all $x \in D$, then T is chaotic on X .

The above conditions are also sufficient for another important notion in linear dynamics, called frequent hypercyclicity, initiated by F. Bayart and S. Grivaux [3]. We refer the reader to [4] and [10] for an introductory discussion of the topic.

Recall that a series $\sum_n u_n$ in a Banach space X is said to be *unconditionally convergent* if $\sum_n u_{\sigma(n)}$ is convergent for all permutations σ on \mathbb{N} . It is well known that the unconditional convergence of $\sum_n u_n$ is equivalent to the following: for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left\| \sum_{n \in F} u_n \right\| < \epsilon,$$

for all finite subsets F of $\{N, N+1, N+2, \dots\}$ (see, for instance [4], page 138).

The following version of Chaoticity Criterion will be useful in what follows. This is probably known to experts, though we cannot find an exact reference (however, see Remark 9.10 in [19] and page 386 in [10]).

Theorem 2.2. (*Chaoticity Criterion*) *Let X be a Banach space, $T \in \mathcal{B}(X)$ and D be a dense set. Then T is chaotic if for each $x \in D$, there exists a sequence $\{u_k\}_{k \geq 0}$ in X with*

$u_0 = x$ such that

$$\sum_{n \geq 0} T^n x \quad \text{and} \quad \sum_{n \geq 0} u_n,$$

are unconditionally convergent, and

$$T^n u_k = u_{k-n},$$

for all $k \geq n$.

Proof. Observe that if $x \in D$, then

$$T^n x \rightarrow 0 \text{ and } f_n := u_n \rightarrow 0,$$

as $n \rightarrow \infty$. Moreover,

$$T^n f_n = T^n u_n = u_{n-n} = u_0 = x,$$

for all $n \geq 0$, and hence, in particular

$$T^n f_n \rightarrow x,$$

as $n \rightarrow \infty$. This shows that T satisfies the Hypercyclicity Criterion with respect to the same dense set D and the sequence $n_k = k$ for all $k \in \mathbb{N}$, and hence T is hypercyclic. It remains to show that T has a dense set of periodic points in X . Define

$$x_p = \sum_{n \geq 1} T^{np} x + x + \sum_{n \geq 1} u_{np},$$

for all $p \geq 1$. By the unconditional convergence of the given series we have that $x_p \rightarrow x$, as $p \rightarrow \infty$. On the other hand

$$T^p x_p = \left(\sum_{n \geq 1} T^{(n+1)p} u_{np} \right) + T^p x + \left(\sum_{n \geq 1} T^p u_{np} \right)$$

together with the properties of $\{u_n\}$ gives

$$T^p(x_p) = x_p,$$

that is, x_p is a periodic point for T . The result now follows from the fact that D is dense in X and $\{x_p\}$ approximates the element x in D . \square

For our purposes (cf. Theorem 4.2) it is also relevant to recall the complete characterization of chaoticity for backward weighted shifts B_λ on l^p spaces (see Grosse-Erdmann [18]): B_λ on l^p is chaotic if and only if

$$\sum_n (\lambda_1 \lambda_2 \cdots \lambda_n)^{-p} < \infty.$$

The case $p = 2$ yields that M_z^* on $H^2(\beta)$ is chaotic if and only if

$$\sum_n \beta_n^2 < \infty.$$

3. REPRODUCING KERNEL HILBERT SPACES AND DERIVATIVES

We first recall the basics and constructions of reproducing kernel Hilbert spaces [1]. Let \mathcal{E} be a Hilbert space. An operator-valued function $K : \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{B}(\mathcal{E})$ is called an *analytic kernel* (cf. [21]) if K is analytic in the first variable and

$$\sum_{i,j=1}^n \langle K(z_i, z_j) \eta_j, \eta_i \rangle_{\mathcal{E}} \geq 0,$$

for all $\{z_i\}_{i=1}^n \subseteq \mathbb{D}$ and $\{\eta_i\}_{i=1}^n \subseteq \mathcal{E}$ and $n \in \mathbb{N}$. In this case there exists a Hilbert space $\mathcal{H}_{\mathcal{E}}(K)$ of \mathcal{E} -valued analytic functions on \mathbb{D} such that

$$\{K(\cdot, w)\eta : w \in \mathbb{D}, \eta \in \mathcal{E}\},$$

is a total set in $\mathcal{H}_{\mathcal{E}}(K)$. Here for $w \in \mathbb{D}$ and $\eta \in \mathcal{E}$, the symbol $K(\cdot, w)\eta$ represents the function

$$(K(\cdot, w)\eta)(z) = K(z, w)\eta,$$

for all $z \in \mathbb{D}$. It is easy to verify that

$$\langle f, K(\cdot, w)\eta \rangle_{\mathcal{H}_{\mathcal{E}}(K)} = \langle f(w), \eta \rangle_{\mathcal{E}},$$

for all $f \in \mathcal{H}_{\mathcal{E}}(K)$, $w \in \mathbb{D}$ and $\eta \in \mathcal{E}$ (cf. [21]). In other words

$$\langle ev_w f, \eta \rangle_{\mathcal{E}} = \langle f, K(\cdot, w)\eta \rangle_{\mathcal{H}_{\mathcal{E}}(K)}, \quad (3.1)$$

where $ev_w : \mathcal{H}_{\mathcal{E}}(K) \rightarrow \mathcal{E}$ is the (bounded) evaluation map defined by

$$ev_w f = f(w),$$

for all $w \in \mathbb{D}$ and $f \in \mathcal{H}_{\mathcal{E}}(K)$. We call the Hilbert space $\mathcal{H}_{\mathcal{E}}(K)$ the *analytic reproducing kernel Hilbert space* corresponding to the kernel K .

It also follows from (3.1) that the functions

$$w \mapsto ev_w \in \mathcal{B}(\mathcal{H}_{\mathcal{E}}(K), \mathcal{E}) \quad \text{and} \quad w \mapsto K(z, w) \in \mathcal{B}(\mathcal{E}),$$

are analytic and co-analytic on \mathbb{D} , respectively, and

$$K(z, w)^* = K(w, z),$$

and

$$ev_z \circ ev_w^* = K(z, w),$$

for all $z, w \in \mathbb{D}$. Furthermore, note that

$$\begin{aligned} \|K(\cdot, w)\eta\|_{\mathcal{H}_{\mathcal{E}}(K)}^2 &= \langle K(\cdot, w)\eta, K(\cdot, w)\eta \rangle_{\mathcal{H}_{\mathcal{E}}(K)} \\ &= \langle K(w, w)\eta, \eta \rangle_{\mathcal{E}}, \end{aligned}$$

that is

$$\|K(\cdot, w)\eta\|_{\mathcal{H}_{\mathcal{E}}(K)} = \|K(w, w)^{1/2}\eta\|_{\mathcal{E}}, \quad (3.2)$$

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{E}$.

Now let $\mathcal{H}_{\mathcal{E}}(K)$ be an analytic reproducing kernel Hilbert space. If $z_0 \in \mathbb{D}$ and $f \in \mathcal{H}_{\mathcal{E}}(K)$, then

$$\begin{aligned} \left(\frac{ev_{z_0+h} - ev_{z_0}}{h} \right) f &= \frac{f(z_0+h) - f(z_0)}{h} \\ &\rightarrow f'(z_0), \end{aligned}$$

as $h \rightarrow 0$. This along with the closed graph theorem implies that $\frac{\partial ev_z}{\partial z} \Big|_{z=z_0} : \mathcal{H}_\mathcal{E}(K) \rightarrow \mathcal{E}$ is a bounded linear operator for each $z_0 \in \mathbb{D}$ and

$$\frac{\partial ev_z}{\partial z} \Big|_{z=z_0} (f) = f'(z_0),$$

for all $f \in \mathcal{H}_\mathcal{E}(K)$ and $z_0 \in \mathbb{D}$. This also yields that the n -th derivative $\frac{\partial^n ev_z}{\partial z^n}$ ($n \in \mathbb{Z}_+$) exists and the bounded linear operator $\frac{\partial^n ev_z}{\partial z^n} \Big|_{z=z_0} : \mathcal{H}_\mathcal{E}(K) \rightarrow \mathcal{E}$ satisfies

$$\frac{\partial^n ev_z}{\partial z^n} \Big|_{z=z_0} (f) = f^{(n)}(z_0),$$

for all $f \in \mathcal{H}_\mathcal{E}(K)$ and $z_0 \in D$. On the other hand, using the identity

$$ev_\lambda^* \eta = K(\cdot, \lambda) \eta,$$

for all $\lambda \in \mathbb{D}$ and $\eta \in \mathcal{E}$, we find, for each $w_0 \in \mathbb{D}$, that

$$\begin{aligned} \left(\frac{ev_{w_0+h}^* - ev_{w_0}^*}{h} \right) \eta &= \frac{K(\cdot, w_0 + h) \eta - K(\cdot, w_0) \eta}{h} \\ &\rightarrow \left(\frac{\partial K}{\partial \bar{w}}(\cdot, w_0) \right) \eta, \end{aligned}$$

as $h \rightarrow 0$, and hence, again by the closed graph theorem, it follows that $\frac{\partial ev_w^*}{\partial \bar{w}} \Big|_{w=w_0} : \mathcal{E} \rightarrow \mathcal{H}_\mathcal{E}(K)$ is a bounded linear operator and

$$\frac{\partial ev_w^*}{\partial \bar{w}} \Big|_{w=w_0} (\eta) = \left(\frac{\partial K}{\partial \bar{w}}(\cdot, w_0) \right) \eta,$$

for all $w_0 \in \mathbb{D}$ and $\eta \in \mathcal{E}$. Hence, $\frac{\partial^n ev_w^*}{\partial \bar{w}^n} \Big|_{w=w_0} : \mathcal{E} \rightarrow \mathcal{H}_\mathcal{E}(K)$, $n \in \mathbb{Z}_+$, is a bounded linear operator, and

$$\frac{\partial^n ev_w^*}{\partial \bar{w}^n} \Big|_{w=w_0} (\eta) = \left(\frac{\partial^n K}{\partial \bar{w}^n}(\cdot, w_0) \right) \eta.$$

Moreover, we have

$$\begin{aligned} \frac{\partial K(\cdot, w)}{\partial \bar{w}} \Big|_{w=w_0} (\eta) &= \lim_{h \rightarrow 0} \frac{ev_{w_0+h}^*(\eta) - ev_{w_0}^*(\eta)}{h} \\ &= \left(\frac{\partial ev_z}{\partial z} \Big|_{z=w_0} \right)^* \eta, \end{aligned}$$

and similarly,

$$\frac{\partial^n K(\cdot, w)}{\partial \bar{w}^n} \Big|_{w=w_0} (\eta) = \left(\frac{\partial^n ev_z}{\partial z^n} \Big|_{z=w_0} \right)^* (\eta), \quad (3.3)$$

for all $\eta \in \mathcal{E}$ and $n \in \mathbb{Z}_+$. Furthermore, for all $f \in \mathcal{H}_\mathcal{E}(K)$, $n \in \mathbb{Z}_+$, $w_0 \in \mathbb{D}$ and $\eta \in \mathcal{E}$, it follows that

$$\begin{aligned} \left\langle f, \left(\frac{\partial^n K}{\partial \bar{w}^n}(\cdot, w) \Big|_{w=w_0} \right) \eta \right\rangle_{\mathcal{H}_\mathcal{E}(K)} &= \frac{\partial^n}{\partial \bar{w}^n} \left(\left\langle f, K(\cdot, w) \eta \right\rangle_{\mathcal{H}_\mathcal{E}(K)} \right) \Big|_{w=w_0} \\ &= \frac{\partial^n}{\partial \bar{w}^n} \left(\left\langle f(w), \eta \right\rangle_{\mathcal{E}} \right) \Big|_{w=w_0} \\ &= \left\langle f^{(n)}(w_0), \eta \right\rangle_{\mathcal{E}}, \end{aligned}$$

that is

$$\langle f^{(n)}(w_0), \eta \rangle_{\mathcal{E}} = \left\langle f, \frac{\partial^n e v_w^*}{\partial \bar{w}^n} \Big|_{w=w_0} \eta \right\rangle_{\mathcal{H}_{\mathcal{E}}(K)}.$$

In particular, if $n \in \mathbb{Z}_+$ and $\eta \in \mathcal{E}$, then (3.3) implies that the function

$$z \mapsto \left(\frac{\partial^n K}{\partial \bar{w}^n}(z, 0) \right) \eta,$$

is in $\mathcal{H}_{\mathcal{E}}(K)$. Therefore, we have proved the following:

Theorem 3.1. *Let $\mathcal{H}_{\mathcal{E}}(K)$ be an analytic reproducing kernel Hilbert space. Let*

$$K_n(z) = \frac{\partial^n K}{\partial \bar{w}^n}(z, 0),$$

and suppose that

$$K_{n,\eta}(z) = K_n(z)\eta,$$

for all $z \in \mathbb{D}$, $\eta \in \mathcal{E}$ and $n \in \mathbb{Z}_+$. Then $K_n(z) \in \mathcal{B}(\mathcal{E})$ for all $z \in \mathbb{D}$, $K_{n,\eta} \in \mathcal{H}_{\mathcal{E}}(K)$ and

$$\langle f^{(n)}(0), \eta \rangle_{\mathcal{E}} = \langle f, K_{n,\eta} \rangle_{\mathcal{H}_{\mathcal{E}}(K)},$$

for all $f \in \mathcal{H}_{\mathcal{E}}(K)$, $\eta \in \mathcal{E}$ and $n \in \mathbb{Z}_+$.

The following formulae for the inner product and norm are now immediate:

Corollary 3.2. *In the setting of Theorem 3.1, we have*

$$\langle K_{m,\zeta}, K_{n,\eta} \rangle_{\mathcal{H}_{\mathcal{E}}(K)} = \left\langle \left(\frac{\partial^{n+m} K}{\partial z^n \partial \bar{w}^m}(0, 0) \right) \zeta, \eta \right\rangle_{\mathcal{E}},$$

for all $m, n \in \mathbb{Z}_+$ and $\eta, \zeta \in \mathcal{E}$. In particular, we have

$$\|K_{n,\eta}\|_{\mathcal{H}_{\mathcal{E}}(K)}^2 = \left\langle \left(\frac{\partial^{2n} K}{\partial z^n \partial \bar{w}^n}(0, 0) \right) \eta, \eta \right\rangle_{\mathcal{E}}.$$

Proof. By using the identity in Theorem 3.1 with $f = K_{m,\zeta}$, the result follows. \square

For example, in the particular case of scalar-valued kernel K , we have

$$K(z, w) = \sum_{m,n \geq 0} a_{mn} z^m \bar{w}^n,$$

for some (not necessarily bounded) infinite matrix (a_{mn}) . Then the inner product and norm expressions in Corollary 3.2 are given by $n!m!a_{mn}$ and $n!\sqrt{a_{nn}}$, respectively.

Theorem 3.1 and Corollary 3.2 should be compared with Lemmas 4.1 and 4.3 in [13] on generalized Bergman kernels.

Now let M_z denote the multiplication operator on $\mathcal{H}_{\mathcal{E}}(K)$, that is

$$(M_z f)(w) = w f(w),$$

for all $f \in \mathcal{H}_{\mathcal{E}}(K)$ and $w \in \mathbb{D}$. If M_z is bounded, $w \in \mathbb{D}$, $\eta \in \mathcal{E}$ and $f \in \mathcal{H}_{\mathcal{E}}(K)$, then

$$\begin{aligned} \langle M_z^*(K(\cdot, w)\eta), f \rangle_{\mathcal{H}_{\mathcal{E}}(K)} &= \langle K(\cdot, w)\eta, z f \rangle_{\mathcal{H}_{\mathcal{E}}(K)} \\ &= \langle \eta, w f(w) \rangle_{\mathcal{E}} \\ &= \langle \bar{w}\eta, f(w) \rangle_{\mathcal{E}} \\ &= \langle \bar{w}K(\cdot, w)\eta, f \rangle_{\mathcal{H}_{\mathcal{E}}(K)}, \end{aligned}$$

that is

$$M_z^*(K(\cdot, w)\eta) = \bar{w}K(\cdot, w)\eta. \quad (3.4)$$

In particular

$$M_z^*(K(\cdot, 0)\eta) = 0, \quad (3.5)$$

for all $\eta \in \mathcal{E}$. Now using

$$K(\cdot, w)\eta = \sum_{n=0}^{\infty} \frac{1}{n!} K_{n,\eta} \bar{w}^n,$$

it follows from (3.4) that

$$M_z^*\left(\sum_{n=0}^{\infty} \frac{1}{n!} K_{n,\eta} \bar{w}^n\right) = \bar{w} \sum_{n=0}^{\infty} \frac{1}{n!} K_{n,\eta} \bar{w}^n,$$

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{E}$. Equating the coefficients of the same powers of \bar{w} on either side, we immediately get the following ‘‘backward shift’’ property of M_z^* :

Theorem 3.3. *Let $\mathcal{H}_{\mathcal{E}}(K)$ be an analytic reproducing kernel Hilbert space. If the multiplication operator M_z is bounded on $\mathcal{H}_{\mathcal{E}}(K)$, then*

$$M_z^*\left(\frac{1}{n!} K_{n,\eta}\right) = \begin{cases} \frac{1}{(n-1)!} K_{n-1,\eta} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0, \end{cases}$$

for all $\eta \in \mathcal{E}$.

4. DYNAMICS OF M_z^*

In this section we present the main results on dynamics of M_z^* on $\mathcal{H}_{\mathcal{E}}(K)$. Our first result concerns hypercyclicity and topological mixing. The second result is about the chaoticity of M_z^* .

Theorem 4.1. *Let \mathcal{E} be a Hilbert space, \mathcal{E}_0 a dense subset of \mathcal{E} and let $\mathcal{H}_{\mathcal{E}}(K)$ be an analytic reproducing kernel Hilbert space. If M_z is bounded on $\mathcal{H}_{\mathcal{E}}(K)$, then the following hold:*

(1) *If*

$$\liminf_n \left(\frac{1}{(n!)^2} \left\langle \left(\frac{\partial^{2n} K}{\partial z^n \partial \bar{w}^n}(0, 0) \right) \eta, \eta \right\rangle_{\mathcal{E}} \right) = 0,$$

uniformly in $\eta \in \mathcal{E}_0$, then M_z^ is hypercyclic.*

(2) *If*

$$\lim_n \left(\frac{1}{(n!)^2} \left\langle \left(\frac{\partial^{2n} K}{\partial z^n \partial \bar{w}^n}(0, 0) \right) \eta, \eta \right\rangle_{\mathcal{E}} \right) = 0,$$

for all $\eta \in \mathcal{E}_0$, then M_z^ is topologically mixing.*

Proof. We apply Theorem 2.1, the Hypercyclicity Criterion, to M_z^* . For each $\eta \in \mathcal{E}_0$ and $m \in \mathbb{Z}_+$, define $\hat{K}_{m,\eta} \in \mathcal{H}_{\mathcal{E}}(K)$ (see Theorem 3.1) by

$$\hat{K}_{m,\eta}(z) = \frac{1}{m!} \left(\frac{\partial^m K}{\partial \bar{w}^m}(z, 0) \right) \eta.$$

Then the set D is dense in $\mathcal{H}_{\mathcal{E}}(K)$, where

$$D = \text{span}\{\hat{K}_{m,\eta} : m \in \mathbb{Z}_+, \eta \in \mathcal{E}_0\}.$$

Indeed, if $g \in \mathcal{H}_{\mathcal{E}}(K)$ is orthogonal to D , then

$$\langle g, \hat{K}_{m,\eta} \rangle = 0,$$

and so, by Theorem 3.1, we have

$$\langle g^{(m)}(0), \eta \rangle_{\mathcal{E}} = 0,$$

for all $m \in \mathbb{Z}_+$ and $\eta \in \mathcal{E}_0$. Since \mathcal{E}_0 is dense in \mathcal{E} , it follows that

$$g^{(m)}(0) = 0,$$

for all $m \in \mathbb{Z}_+$ and hence

$$g \equiv 0.$$

Now note that $\{\hat{K}_{m,\eta} : m \in \mathbb{Z}_+, \eta \in \mathcal{E}_0\}$ is a generating set for D . If $\zeta \in \mathcal{E}$, $s > t$ and $s, t \in \mathbb{Z}_+$, then by (3.5) and Theorem 3.3, it follows that

$$M_z^{*s} \hat{K}_{t,\zeta} = 0. \quad (4.1)$$

Let

$$f = \sum_{j=1}^p \alpha_j \hat{K}_{m_j, \eta_j} \in D,$$

for some positive integer p , $\alpha_j \in \mathbb{C}$, $m_j \in \mathbb{Z}_+$ and $\eta_j \in \mathcal{E}_0$, $1 \leq j \leq p$. Then (4.1) implies, in particular, that

$$M_z^{*n} f \rightarrow 0,$$

as $n \rightarrow \infty$. This proves the first condition of the criterion.

To prove the second condition of the criterion, first we define a sequence $\{g_n\} \subseteq D$ by

$$g_n := \sum_{j=1}^p \alpha_j \hat{K}_{m_j+n, \eta_j},$$

for all $n \in \mathbb{N}$. By Theorem 3.3, it follows that

$$M_z^{*n}(\hat{K}_{m_j+n, \eta_j}) = \hat{K}_{m_j, \eta_j},$$

for $j = 1, \dots, p$, and hence

$$M_z^{*n} g_n = f, \quad (4.2)$$

for $n \in \mathbb{N}$. Since

$$\|\hat{K}_{m,\eta}\|_{\mathcal{H}_{\mathcal{E}}(K)}^2 = \frac{1}{(m!)^2} \left\langle \left(\frac{\partial^{2m} K}{\partial z^m \partial \bar{w}^m}(0,0) \right) \eta, \eta \right\rangle_{\mathcal{E}},$$

for all $m \geq 0$, by Corollary 3.2, and

$$\liminf_n \left(\frac{1}{(n!)^2} \left\langle \left(\frac{\partial^{2n} K}{\partial z^n \partial \bar{w}^n}(0,0) \right) \eta, \eta \right\rangle_{\mathcal{E}} \right) = 0,$$

uniformly in $\eta \in \mathcal{E}_0$ by assumption, it follows that there exists a sequence of natural numbers $\{t_k\}$ such that

$$\hat{K}_{t_k, \eta} \rightarrow 0,$$

in $\mathcal{H}_{\mathcal{E}}(K)$, for all $\eta \in \mathcal{E}_0$. This implies that

$$M_z^{*j}(\hat{K}_{t_k, \eta}) \rightarrow 0,$$

as $k \rightarrow \infty$ and for all $j \geq 0$. Now observe that

$$\hat{K}_{t_k-j, \eta} = M_z^{*j}(\hat{K}_{t_k, \eta}),$$

for all $t_k \geq j$. Hence

$$\hat{K}_{t_k-j, \eta} \rightarrow 0,$$

as $k \rightarrow \infty$ and for all $j \geq 0$.

We now use the following lemma (see Lemma 4.2, page 90, [19]): Let (X, d) be a metric space and let $\{y_k\}$ be a sequence in X . Suppose that $\{t_k\}$ is a sequence of natural numbers. If the subsequence $\{y_{t_k-j}\}$ converges to a fixed element y for each j , then there exists $\{n_k\}$ such that $\{y_{n_k+j}\}$ converges to y .

This shows that there exists a strictly increasing sequence of natural numbers $\{n_k\}_k$ such that

$$\hat{K}_{n_k+j, \eta} \rightarrow 0,$$

for each j and for all $\eta \in \mathcal{E}_0$. Set

$$f_k := g_{n_k},$$

that is

$$f_k = \sum_{j=1}^p \alpha_j \hat{K}_{n_k+m_j, \eta_j},$$

for all $k \geq 1$. Clearly

$$f_k \rightarrow 0,$$

as $k \rightarrow \infty$. Moreover, by (4.2) we have

$$M_z^{*n_k} f_k \rightarrow f,$$

as $k \rightarrow \infty$. Hence M_z^* satisfies the Hypercyclicity Criterion with respect to $\{n_k\}$. This proves (1).

For the second part, we proceed with D and f as in the proof of part (1) above. In this case, however, by assumption, it follows that

$$\hat{K}_{n, \eta} \rightarrow 0,$$

as $n \rightarrow \infty$ and for all $\eta \in \mathcal{E}_0$. If we set

$$f_n = \sum_{j=1}^p \alpha_j \hat{K}_{n+m_j, \eta_j},$$

for all $n \geq 1$, then, as in the proof of part (1), it follows that

$$f_n \rightarrow 0 \quad \text{and} \quad M_z^{*n} f_n \rightarrow f,$$

as $n \rightarrow \infty$. This concludes the proof. \square

We now proceed to the chaos of M_z^* on $\mathcal{H}_{\mathcal{E}}(K)$. Recall that a double series

$$\sum_{i, j \geq 0} u_{i, j},$$

in a Hilbert space \mathcal{H} is said to be convergent if the sequence $\{u_n\}$ is convergent in \mathcal{H} , where

$$u_n = \sum_{i, j \leq n} u_{i, j},$$

for all $n \geq 0$.

Theorem 4.2. *Let \mathcal{E} be a Hilbert space, \mathcal{E}_0 a dense subset of \mathcal{E} and let $\mathcal{H}_{\mathcal{E}}(K)$ be an analytic reproducing kernel Hilbert space. If M_z is bounded on $\mathcal{H}_{\mathcal{E}}(K)$ and the double series*

$$\sum_{n,m \geq 0} \frac{1}{n! m!} \left\langle \left(\frac{\partial^{n+m} K}{\partial z^n \partial \bar{w}^m}(0,0) \right) \eta, \eta \right\rangle_{\mathcal{E}},$$

is absolutely convergent for all $\eta \in \mathcal{E}_0$, then M_z^ chaotic.*

Proof. Here we apply Theorem 2.2, the Chaoticity Criterion, and proceed with the set D as defined in the proof of Theorem 4.1. Assume that

$$f := \sum_{j=1}^p \alpha_j \hat{K}_{m_j, \eta_j} \in D,$$

and choose

$$u_k := \sum_{j=1}^p \alpha_j \hat{K}_{k+m_j, \eta_j}.$$

From the proof of Theorem 4.1 it follows that for $k \geq n$,

$$M_z^{*n} u_k = \sum_{j=1}^p \alpha_j \hat{K}_{k-n+m_j, \eta_j} = u_{k-n}.$$

Now using (4.1) we obtain that the series

$$\sum_{n \geq 0} M_z^{*n} \hat{K}_{m, \eta},$$

is unconditionally convergent in $\mathcal{H}_{\mathcal{E}}(K)$ for each $m \in \mathbb{Z}_+$ and $\eta \in \mathcal{E}_0$, and so

$$\sum_{n \geq 0} M_z^{*n} f,$$

is also unconditionally convergent in $\mathcal{H}_{\mathcal{E}}(K)$ for all $f \in D$. Now, we verify that the series $\sum_n u_n$ is unconditionally convergent. It is, however, enough to check the above for

$$f = \hat{K}_{m, \eta},$$

where $m \in \mathbb{Z}_+$ and $\eta \in \mathcal{E}_0$, that is, we need to verify that the series

$$\sum_{n \geq 0} \hat{K}_{m+n, \eta}.$$

is unconditionally convergent in $\mathcal{H}_{\mathcal{E}}(K)$. Indeed, for $\epsilon > 0$ and $\eta \in \mathcal{E}_0$, by hypothesis, there exists $N \in \mathbb{N}$ such that

$$\sum_{n,m \in F} \frac{1}{n! m!} \left\langle \left(\frac{\partial^{n+m} K}{\partial z^n \partial \bar{w}^m}(0,0) \right) \eta, \eta \right\rangle_{\mathcal{E}} < \epsilon,$$

for all finite sets $F \subset [N, \infty) \cap \mathbb{N}$. Then Corollary 3.2 implies that

$$\left\langle \sum_{n \in F} \hat{K}_{n, \eta}, \sum_{m \in F} \hat{K}_{m, \eta} \right\rangle < \epsilon,$$

that is

$$\left\| \sum_{n \in F} \hat{K}_{n,\eta} \right\|_{\mathcal{H}_{\mathcal{E}}(K)} < \sqrt{\epsilon}.$$

Hence the series $\sum_{n \geq 0} \hat{K}_{n,\eta}$ is unconditionally convergent in $\mathcal{H}_{\mathcal{E}}(K)$ (see the discussion preceding Theorem 2.2). This completes the proof of the theorem. \square

Of particular interest is the case where K is a scalar-valued kernel. Here the sufficient condition for dynamics of M_z^* involves the diagonal of the kernel function. We will address this issue in Theorem 5.1.

The converse of Theorem 4.1 is not true in general (see the example in Subsection 5.3 and also Section 6).

5. EXAMPLES AND APPLICATIONS

In this section we give concrete examples and applications of our main results.

5.1. Weighted shifts. We begin by noting that the weighted shift space $H^2(\beta)$, as introduced in Section 2, is an analytic reproducing kernel Hilbert space. In this case, the scalar-valued reproducing kernel function is given by

$$K(z, w) = \sum_{n \geq 0} \beta_n^2 z^n \bar{w}^n,$$

for all $z, w \in \mathbb{D}$. Moreover

$$\frac{\partial^{n+m} K}{\partial z^n \bar{w}^m}(0, 0) = \begin{cases} (n!)^2 \beta_n^2 & \text{if } m = n \\ 0 & \text{if } m \neq n, \end{cases}$$

and $m, n \geq 0$. By Theorem 4.1, it follows that M_z^* is hypercyclic on $H^2(\beta)$ if

$$\liminf_n \beta_n = 0.$$

This condition is also necessary for hypercyclicity of M_z^* on $H^2(\beta)$ (see Salas [25]) but not in general on $\mathcal{H}_{\mathcal{E}}(K)$ (see the Subsection 5.3). A similar classification result also holds for chaoticity and mixing for M_z^* on $H^2(\beta)$ (cf. [19]).

5.2. Quasi-scalar reproducing kernel Hilbert spaces. We say that a kernel function $K : \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{B}(\mathcal{E})$ is a quasi-scalar kernel if there exists a scalar-valued analytic kernel $k : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$ such that

$$K(z, w) = k(z, w)I_{\mathcal{E}},$$

for all $z, w \in \mathbb{D}$. In this case, the general construction of reproducing kernel Hilbert spaces yields that (for instance, see [21])

$$\mathcal{H}_{\mathcal{E}}(K) \cong \mathcal{H}(k) \otimes \mathcal{E},$$

where $\mathcal{H}(k)$ is the reproducing kernel Hilbert space corresponding to the kernel function k on \mathbb{D} . Quasi-scalar reproducing kernel Hilbert spaces play important roles in function theory and operator theory, particularly in the study of dilation theory and analytic model theory (cf. [21]).

As a simple example of the use of Theorems 4.1 and 4.2, we prove the following:

Theorem 5.1. *If $\mathcal{H}_K(\mathcal{E})$ is a quasi-scalar reproducing kernel Hilbert space and M_z on $\mathcal{H}_K(\mathcal{E})$ is bounded, then:*

(1) M_z^* is hypercyclic if

$$\liminf_n \frac{1}{(n!)^2} \frac{\partial^{2n} k}{\partial z^n \partial \bar{w}^n}(0, 0) = 0.$$

(2) M_z^* is topologically mixing if

$$\lim_n \frac{1}{(n!)^2} \frac{\partial^{2n} k}{\partial z^n \partial \bar{w}^n}(0, 0) = 0.$$

(3) M_z^* is chaotic if the series

$$\sum_{n,m \geq 0} \frac{1}{n! m!} \frac{\partial^{n+m} k}{\partial z^n \partial \bar{w}^m}(0, 0),$$

is absolutely convergent.

Proof. If $k : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$ is an analytic kernel on \mathbb{D} and

$$K(z, w) = k(z, w)I_{\mathcal{E}},$$

for all $z, w \in \mathbb{D}$, then

$$\|K(\cdot, w)\eta\|_{\mathcal{H}_{\mathcal{E}}(K)} = \|k(\cdot, w)\|_{\mathcal{H}(k)}\|\eta\|_{\mathcal{E}},$$

for all $\eta \in \mathcal{E}$. Observe now that

$$\left\langle \left(\frac{\partial^{2n} K}{\partial z^n \partial \bar{w}^n}(0, 0) \right) \eta, \eta \right\rangle_{\mathcal{E}} = \left(\frac{\partial^{2n} k}{\partial z^n \partial \bar{w}^n}(0, 0) \right) \|\eta\|_{\mathcal{E}}^2,$$

for all $w \in \mathbb{D}$, $\eta \in \mathcal{E}$ and $n \geq 0$. The result now follows from Theorems 4.1 and 4.2. \square

It is worth pointing out that the conclusion of the above theorem is independent of the choice of the Hilbert space \mathcal{E} . More specifically, if k is a scalar kernel and

$$\liminf_n \frac{1}{(n!)^2} \frac{\partial^{2n} k}{\partial z^n \partial \bar{w}^n}(0, 0) = 0,$$

then M_z^* on $\mathcal{H}_{\mathcal{E}}(K)$ is hypercyclic where

$$K(z, w) = k(z, w)I_{\mathcal{E}},$$

for all $z, w \in \mathbb{D}$ and \mathcal{E} is a Hilbert space. This observation also should be compared with the linear dynamics of tensor products of operators (cf. [9] and [23]).

As a concrete application, consider the \mathcal{E} -valued Dirichlet space $\mathcal{D}_{\mathcal{E}}$ on \mathbb{D} , where \mathcal{E} is separable Hilbert space. Notice that the kernel function for $\mathcal{D}_{\mathcal{E}}$ is given by

$$(z, w) \mapsto \left(\sum_{n \geq 0} \frac{z^n \bar{w}^n}{n+1} \right) I_{\mathcal{E}}.$$

From the above theorem, it then follows that M_z^* is mixing on $\mathcal{D}_{\mathcal{E}}$.

5.3. A counter-example. Here we present a counterexample to show that the sufficient condition in Theorem 5.1 for M_z^* to be hypercyclic is not a necessary condition. Consider the Hilbert space $H^2(\beta) \subseteq \mathcal{O}(\mathbb{D})$, as in Section 2, corresponding to the (diagonal) kernel

$$k(z, w) = \sum_{n \geq 0} \beta_n^2 z^n \bar{w}^n \quad (z, w \in \mathbb{D}),$$

where $\{\beta_n\}$ is a sequence of strictly positive real numbers. Suppose that $M_{z,k}$, the multiplication operator by the coordinate function z , on $H^2(\beta)$ is bounded. Let

$$\theta(z) = \frac{1}{1-z},$$

for all $z \in \mathbb{D}$, and set

$$\mathcal{H} = \{\theta f : f \in H^2(\beta)\}.$$

Then \mathcal{H} is a Hilbert space with the inner product

$$\langle \theta f, \theta g \rangle_{\mathcal{H}} := \langle f, g \rangle_{H^2(\beta)},$$

for all $f, g \in H^2(\beta)$. Moreover, \mathcal{H} is an analytic reproducing kernel Hilbert space corresponding to the kernel

$$k_{\theta}(z, w) = \theta(z) \left(\sum_{n \geq 0} \beta_n^2 z^n \bar{w}^n \right) \overline{\theta(w)},$$

for all $z, w \in \mathbb{D}$. Here the reproducing property is given by

$$\langle \theta f, k_{\theta}(\cdot, w) \rangle_{\mathcal{H}} = \langle \theta f, \overline{\theta(w)} \theta k(\cdot, w) \rangle_{\mathcal{H}} = \theta(w) \langle f, k(\cdot, w) \rangle_{H^2(\beta)} = \theta(w) f(w),$$

for all $f \in \mathcal{D}$ and $w \in \mathbb{D}$. Since

$$\{\beta_n z^n\}_{n \geq 0},$$

is an orthonormal basis in $H^2(\beta)$, it follows that

$$\{\beta_n \theta z^n\}_{n \geq 0},$$

forms an orthonormal basis in \mathcal{H} . Also observe that the multiplication operator M_z on \mathcal{H} is a bounded operator. Moreover, it follows that M_z on \mathcal{H} and $M_{z,k}$ on $H^2(\beta)$ are unitarily equivalent. Since the condition

$$\liminf_n \beta_n = 0,$$

is necessary and sufficient for $M_{z,k}^*$ to be hypercyclic on $H^2(\beta)$ [25], under this assumption we conclude that M_z^* is also hypercyclic on \mathcal{H} . However, the sufficient condition for hypercyclicity in Theorem 5.1 (and hence, that of Theorem 4.1) is not satisfied for M_z^* on \mathcal{H} . Indeed, observe that

$$k_{\theta}(z, w) = (1 + z + z^2 + \cdots) (\beta_0^2 + \beta_1^2 z \bar{w} + \beta_2^2 z^2 \bar{w}^2 + \cdots) (1 + \bar{w} + \bar{w}^2 + \cdots),$$

for all $z, w \in \mathbb{D}$. Then a_{nn} , the coefficient of $z^n \bar{w}^n$ in the above expansion, is given by

$$a_{nn} = \beta_0^2 + \beta_1^2 + \cdots + \beta_n^2,$$

for all $n \geq 0$, and so

$$\liminf_n a_{nn} \neq 0.$$

On the other hand

$$a_{nn} = \frac{1}{(n!)^2} \frac{\partial^{2n} k_\theta}{\partial z^n \partial \bar{w}^n}(0, 0)$$

for all $n \geq 0$. This shows that the sufficient condition for hypercyclicity in Theorem 4.1 is not necessary.

The construction above also allows us to work with non-zero polynomials instead of θ . However, in this case, the sufficient condition in Theorem 5.1 for M_z^* to be hypercyclic is also a necessary condition (see Section 6).

5.4. Sums of reproducing kernels. Here we consider scalar-valued analytic kernel functions on \mathbb{D} . To avoid confusion, for an analytic reproducing kernel Hilbert space $\mathcal{H}(k)$, we denote the multiplication operator M_z on $\mathcal{H}(k)$ (provided it is bounded) by $M_{z,k}$.

It is easy to see that if k_1 and k_2 are analytic kernels on \mathbb{D} , then $k_1 + k_2$ is also an analytic kernel on \mathbb{D} . Moreover, it follows from Aronszajn [1] that

$$\mathcal{H}(k_1 + k_2) = \mathcal{H}(k_1) + \mathcal{H}(k_2),$$

and

$$\|f\|_{\mathcal{H}(k_1+k_2)}^2 = \inf\{\|f_1\|_{\mathcal{H}(k_1)}^2 + \|f_2\|_{\mathcal{H}(k_2)}^2 : f = f_1 + f_2, f_j \in \mathcal{H}(k_j), j = 1, 2\},$$

for all $f \in \mathcal{H}(k_1 + k_2)$. Now assume that M_{z,k_j} is bounded on $\mathcal{H}(k_j)$, $j = 1, 2$. It is then evident from the above definition that M_{z,k_1+k_2} is also bounded on $\mathcal{H}(k_1 + k_2)$.

Theorem 5.2. *Let $\mathcal{H}(k_1)$ and $\mathcal{H}(k_2)$ be analytic reproducing kernel Hilbert spaces over \mathbb{D} . If M_{z,k_1} and M_{z,k_2} are bounded, and M_{z,k_1}^* and M_{z,k_2}^* satisfy the condition for hypercyclicity (respectively, mixing and chaoticity) in Theorem 5.1, then $M_{z,k_1+k_2}^*$ is hypercyclic (respectively, mixing and chaotic).*

Proof. Note that

$$\frac{\partial^n(k_1 + k_2)}{\partial \bar{w}^n}(\cdot, 0) = \frac{\partial^n k_1}{\partial \bar{w}^n}(\cdot, 0) + \frac{\partial^n k_2}{\partial \bar{w}^n}(\cdot, 0),$$

and

$$\frac{\partial^n(k_1 + k_2)}{\partial \bar{w}^n}(\cdot, 0) \in \mathcal{H}(k_1 + k_2) \quad \text{and} \quad \frac{\partial^n k_j}{\partial \bar{w}^n}(\cdot, 0) \in \mathcal{H}(k_j),$$

for all $j = 1, 2$ and $n \geq 0$. Hence

$$\left\| \frac{\partial^n(k_1 + k_2)}{\partial \bar{w}^n}(\cdot, 0) \right\|_{\mathcal{H}(k_1+k_2)}^2 \leq \left\| \frac{\partial^n k_1}{\partial \bar{w}^n}(\cdot, 0) \right\|_{\mathcal{H}(k_1)}^2 + \left\| \frac{\partial^n k_2}{\partial \bar{w}^n}(\cdot, 0) \right\|_{\mathcal{H}(k_2)}^2.$$

The result now follows directly from Theorem 5.1. \square

6. NECESSARY CONDITIONS AND CONCLUDING REMARKS

In the setting of analytic reproducing kernel Hilbert spaces many fundamental and basic questions about dynamics remain unanswered. For instance, the converse of Theorem 4.1 is false in general (see the example in Subsection 5.3). Here we show that under certain assumptions on the analytic kernel functions, the sufficient condition for hypercyclicity in Theorem 4.1 is also necessary.

The first construction is based on the setting of the counterexample in Subsection 5.3 but replacing the analytic function θ by (operator-valued) analytic polynomials. Let \mathcal{E} be a Hilbert space and let $P(z)$ be a $\mathcal{B}(\mathcal{E})$ -valued (analytic) polynomial. Suppose that

$$P(z) = A_0 + A_1z + \cdots + A_dz^d,$$

for some $A_j \in \mathcal{B}(\mathcal{E})$, $j = 0, \dots, d$. Now we consider a scalar-valued analytic (and diagonal) kernel k on \mathbb{D} :

$$k(z, w) = \sum_{n \geq 0} \beta_n^2 z^n \bar{w}^n,$$

where β_n are non-negative numbers, and let $\mathcal{H}_{\mathcal{E}}(K)$ denote the reproducing kernel Hilbert space corresponding to the kernel

$$K(z, w) = k(z, w)I_{\mathcal{E}},$$

for all $z, w \in \mathbb{D}$. Let

$$K_P(z, w) = P(z)K(z, w)P(w)^*, \quad (6.1)$$

for all $z, w \in \mathbb{D}$. Then K_P is a $\mathcal{B}(\mathcal{E})$ -valued kernel on \mathbb{D} . Set

$$\mathcal{H}_P = \{Pf : f \in \mathcal{H}_{\mathcal{E}}(K)\},$$

where

$$(Pf)(z) = P(z)f(z),$$

for all $f \in \mathcal{H}_{\mathcal{E}}(K)$ and $z \in \mathbb{D}$. Clearly \mathcal{H}_P is a vector space of \mathcal{E} -valued analytic functions on \mathbb{D} . Now suppose that A_0 is injective. If $f = \sum_{n \geq 0} \eta_n z^n \in \mathcal{H}_{\mathcal{E}}(K)$ and

$$P(z)f(z) = 0, \quad (6.2)$$

for all $z \in \mathbb{D}$, then

$$P(0)f(0) = A_0\eta_0 = 0,$$

implies that $\eta_0 = 0$. Now differentiating (6.2), one gets

$$P'(z)f(z) + P(z)f'(z) = 0,$$

and hence

$$0 = P'(0)f(0) + P(0)f'(0) = A_0\eta_1.$$

We have $\eta_1 = 0$. Continuing in this way we obtain that $f \equiv 0$. It now follows that \mathcal{H} is a Hilbert space with inner product defined by

$$\langle Pf, Pg \rangle_{\mathcal{H}_P} := \langle f, g \rangle_{\mathcal{H}_{\mathcal{E}}(K)},$$

for all f and g in $\mathcal{H}_{\mathcal{E}}(K)$. Also it follows that \mathcal{H}_P is a reproducing kernel Hilbert space corresponding to the kernel K_P and if $M_{z,K}$ (the multiplication operator on $\mathcal{H}_{\mathcal{E}}(K)$ by the coordinate function z) is bounded, then $M_{z,K}$ on $\mathcal{H}_{\mathcal{E}}(K)$ and M_z on \mathcal{H}_P are unitarily equivalent. Moreover, since $M_{z,K}$ on $\mathcal{H}_{\mathcal{E}}(K)$ and $M_{z,k} \otimes I$ on $\mathcal{H}(k) \otimes \mathcal{E}$ are unitarily equivalent, by a result of Martínez-Giménez and Peris (Proposition 1.14 in [23]) and Salas [25], it follows that M_z^* is hypercyclic on \mathcal{H}_P if and only if

$$\liminf_n \beta_n = 0.$$

Now equating the coefficients of $z^n \bar{w}^n$, say C_{nn} , in the expansion of $K_P(z, w)$ in (6.1), we have (as in Subsection 5.3)

$$C_{nn} = A_0 A_0^* \beta_n^2 + A_1 A_1^* \beta_{n-1}^2 + \cdots + A_d A_d^* \beta_{n-d}^2,$$

for all $n \geq d$. Since $M_{z,k}$ is bounded, it follows that

$$\sup_k \frac{\beta_k}{\beta_{k+1}} < \infty,$$

and hence, there exists $M > 0$ such that

$$\|C_{nn}\|_{\mathcal{B}(\mathcal{E})} \leq M \beta_{n-d}^2,$$

for all $n \geq d$. Thus, if M_z^* is hypercyclic on \mathcal{H}_P , then, by the above observation, we see that

$$\liminf_n \|C_{nn}\|_{\mathcal{B}(\mathcal{E})} = 0.$$

Hence, by

$$C_{nn} = \frac{1}{(n!)^2} \frac{\partial^{2n} K_P}{\partial z^n \partial \bar{w}^n}(0, 0),$$

for all $n \geq 0$, it follows that

$$\liminf_n \left(\frac{1}{(n!)^2} \left\langle \left(\frac{\partial^{2n} K_P}{\partial z^n \partial \bar{w}^n}(0, 0) \right) \eta, \eta \right\rangle_{\mathcal{E}} \right) = 0,$$

uniformly in $\eta \in \mathcal{E}_0$, where $\mathcal{E}_0 = \mathcal{E}$. This proves the converse of the hypercyclicity part in Theorem 4.1. We have therefore shown the following result:

Theorem 6.1. *Let \mathcal{E} be a Hilbert space, k on \mathbb{D} be a scalar-valued kernel and*

$$k(z, w) = \sum_{n \geq 0} \beta_n^2 z^n \bar{w}^n,$$

where β_n are non-negative numbers, and let $\mathcal{H}_{\mathcal{E}}(K)$ denote the reproducing kernel Hilbert space corresponding to the kernel

$$K(z, w) = k(z, w) I_{\mathcal{E}},$$

for all $z, w \in \mathbb{D}$. Suppose that $\{A_j\}_{j=0}^d \subseteq \mathcal{B}(\mathcal{E})$, A_0 is injective,

$$P(z) = A_0 + A_1 z + \cdots + A_d z^d,$$

and let \mathcal{H}_P denote the reproducing kernel Hilbert space corresponding to the kernel function

$$K_P(z, w) = P(z) K(z, w) P(w)^* \quad (z, w \in \mathbb{D}).$$

If M_z is bounded on \mathcal{H}_P , then the following are equivalent:

- (i) M_z^* is hypercyclic.
- (ii) $\liminf_n \left(\frac{1}{(n!)^2} \left\langle \left(\frac{\partial^{2n} K_P}{\partial z^n \partial \bar{w}^n}(0, 0) \right) \eta, \eta \right\rangle_{\mathcal{E}} \right) = 0$ uniformly on $\eta \in \mathcal{E}$.
- (iii) $\liminf_n \beta_n = 0$.

The same method of proof also applies to scalar-valued reproducing kernel Hilbert space \mathcal{H}_P corresponding to the kernel function

$$k_P(z, w) = P(z)k(z, w)\overline{P(w)} \quad (z, w \in \mathbb{D}),$$

where

$$P(z) = a_0 + a_1z + \cdots + a_dz^d,$$

be a *non-zero* (instead of $a_0 \neq 0$) scalar polynomial and k is defined as in Theorem 6.1 (observe that $k_P(z, w)$ is not a diagonal kernel in general). Here $\mathcal{H}_P = \{Pf : f \in \mathcal{H}(k)\}$ and the inner product is given by $\langle Pf, Pg \rangle_{\mathcal{H}_P} = \langle f, g \rangle_{\mathcal{H}(k)}$ for all $f, g \in \mathcal{H}(k)$.

Theorem 6.2. *If M_z is bounded on \mathcal{H}_P , then the following are equivalent:*

- (i) M_z^* is hypercyclic.
- (ii) $\liminf_n \frac{1}{(n!)^2} \left(\frac{\partial^{2n} k}{\partial z^n \partial \bar{w}^n}(0, 0) \right) = 0$.
- (iii) $\liminf_n \beta_n = 0$.

The converse of Theorem 4.1 is also evidently true if the derivatives

$$\left\{ \frac{\partial^m k}{\partial \bar{w}^m}(\cdot, 0) \right\}_{m \in \mathbb{Z}_+},$$

of a scalar-valued kernel k are orthogonal vectors in $\mathcal{H}(k)$. Indeed, if

$$k(z, w) = \sum_{m, n \geq 0} a_{mn} z^m \bar{w}^n,$$

for a_{mn} in \mathbb{C} , then

$$\begin{aligned} a_{mn} &= \frac{1}{m!n!} \frac{\partial^{m+n} k}{\partial z^m \partial \bar{w}^n}(0, 0) \\ &= \frac{1}{m!n!} \left\langle \frac{\partial^n k}{\partial \bar{w}^n}(\cdot, 0), \frac{\partial^m k}{\partial \bar{w}^m}(\cdot, 0) \right\rangle_{\mathcal{H}(k)} \\ &= \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{(n!)^2} \frac{\partial^{2n} k}{\partial z^n \partial \bar{w}^n}(0, 0) & \text{if } m = n, \end{cases} \end{aligned}$$

and so

$$k(z, w) = \sum_{n \geq 0} a_{nn} z^n \bar{w}^n,$$

for all $z, w \in \mathbb{D}$. Then M_z^* is a backward weighted shift operator on $\mathcal{H}(k)$. If we now assume that M_z^* is hypercyclic, then by Salas [25], it follows that

$$\begin{aligned} \liminf_n a_{nn} &= \liminf_n \frac{1}{(n!)^2} \frac{\partial^{2n} k}{\partial z^n \partial \bar{w}^n}(0, 0) \\ &= 0. \end{aligned}$$

Again, similar results hold for topological mixing and chaos of M_z^* .

It would be therefore very interesting to determine the class of vector valued analytic kernel functions for which the converse of Theorem 4.1 holds.

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